Cornel S. Pintea

THE SIZE OF CRITICAL AND TANGENCY SETS

Presa Universitară Clujeană

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Contents

Preface **v**

1 The sets of critical points and critical values **1**
1.1 Manifolds and functions. Critical and regular sets 2
1.1.1 Submanifolds, general subsets and their tangent spaces . 6
1.1.2 Differentiable functions and their critical sets 8
1.1.3 Vector fields and differentiable forms 14
1.1.4 Vanishing and critical sets 18
1.2 The degree of a differentiable function. Applications 19
1.3 Classical results in the regular and critical settings 29
1.3.1 The preimage, the Ehresmann and the Sard theorems . 29
1.3.2 The L-S category and the L-S multiplicity theorem . . . 35

2 Transversality and applications **37**
2.1 Definition and basic results 38
2.2 Regularity and transversality 41
2.3 The method of Lagrange multipliers 43
2.3.1 Lagrange multipliers in the finite dimensional setting . . 43
2.3.2 Lagrange multipliers in the infinite dimensional setting . 53
2.4 Critical and tangency points. Tangency sets 61
2.5 The order of connectedness of some induced homomorphisms . 64

3 Functions with isolated cone-like critical points **73**
3.1 Cone-like critical points 73
3.2 Polynomial functions with isolated critical points 74
3.3 Constructions of fibered links 76
3.3.1 Fibered links in dimensions $(2n, n)$, $n \geq 3$ 77
3.3.2 Fibered links in dimensions $(2n + 1, n)$, $n \geq 2$ 78
3.3.3 Fibered links in dimensions $(2n, k), (2n + 1, k), n \geq k \geq 2$ 79
3.4 Lefschetz fibrations and vanishing cycles 79
3.5 Morse functions 81
3.5.1 Real valued Morse functions. The Morse inequalities . . 81
3.5.2 Circle-valued Morse functions 91

4 The inverse critical point problem 97

4.1 The φ-category of a pair of manifolds. Examples 98

4.1.1 The φ-category of some pairs of surfaces 101

4.1.2 The φ-category of some pairs of connected sums 104

4.2 The Morse-Smale characteristics 107

4.2.1 The real M-S characteristic 108

4.2.2 The circular M-S characteristic 109

4.2.3 The circular M-S characteristic of the compact surfaces 113

4.3 On the homology of some complements 118

4.4 CS^{∞}-non-critical families of fibers 121

5 Maps with finite critical sets 125

5.1 Necessary conditions for finite critical sets 126

5.2 Further pairs of manifolds with finite φ 134

5.2.1 Local models for isolated singularities 134

5.2.2 Cut and paste local models 135

5.2.3 Pairs (M^{2n}, S^{n-k}) with finite φ-category 137

6 Maps with infinite critical sets 141

6.1 The zero codimension case 141

6.1.1 The algebraic φ-category 143

6.2 The higher codimension case 149

6.3 The arbitrary codimension case 155

6.3.1 An approach of topologic flavor 155

6.3.2 An approach of geometric flavor 158

7 The size of the tangency set 165

7.1 The minimum number of characteristic points 166

7.2 The size of tangencies to non-involutive distributions 170

7.3 Generalisation of Derridj's Theorem 172

8 BVP and critical points of appropriate energy functionals 175

8.1 A few BVP and a particular eigenvalue-transmission problem . 175

8.1.1 The eigenvalue-transmission problem 177

8.1.2 The variational counterpart of the E-T problem 178

8.1.3 Appropriate energy functionals and their properties . . 181

8.1.4 Infinitely many eigenpairs via the L-S principle 185

8.2 E-T problem with Robin boundary conditions 186

8.3 E-T problems in the Riemannian setting 188

Appendices

A Banach spaces and differential calculus **195**

A.1 The weak topology 195

A.2 The Fréchet and Gâteaux differentiability 196

A.3 The matrix representation of the Fréchet differential 197

A.4 Examples C^1-smooth functionals on Banach spaces 198

A.5 Critical points of differentiable maps 201

B Various dimensions of a space **205**

B.1 The inductive dimensions and the covering dimension 205

B.2 The Hausdorff dimension 208

C Homotopy and (co)homology groups **211**

C.1 Homotopy groups 211

C.2 The singular homology 218

C.3 The de Rham cohomology 225

Bibliography **227**

Preface

We emphasize several faces of the critical phenomena for maps between two manifolds M^m and N^n. In the finite dimensional setting, it basically concerns those points of the source manifold where the rank of the differential is strictly less than its maximum possible value $\min\{m, n\}$. Note that for $n = 1$ the critical points are those where the differential vanishes and this fact is being considered as the definition of the critical points of real valued functions defined on infinite dimensional Banach spaces. The set of images of such points is also part of the critical phenomena and the size of the critical sets and the sets of critical values is sometimes evaluated. One such face concerns the cell structure, up to a homotopy equivalence, of a compact manifold through Morse functions on that manifold. Another face concerns the solution set of a certain *boundary value problem* (shortly abbreviated by BVP) which is expressed as the critical set of a suitable C^1-smooth functional. The size of the critical sets is an important matter in both cases and it is another face of the critical phenomena together with the size of the set of critical values. The latter one is first evaluated through the Sard Theorem. For maps with higher dimensional target manifolds we basically evaluate the size of critical sets. Depending on the maps we are working with, the evaluation tools are the *measure*, the *cardinality* or the *dimension*. Note however that the Sard Theorem works for every type of, sufficiently regular, maps. For the technical details, the involved tools we often use are the *degree* of a map, the *Euler-Poincaré characteristic*, the *homotopy* and the *(co)homology* groups. A connection of critical points with characteristic points is also pointed out and the size of tangency sets is evaluated. The latter evaluation result allows us to generalize the Derridj theorem.

The work is based on published papers by the author alone or jointly and covers all topics of the author's habilitation thesis, but not all results and proofs are included. On the other hand some other topics and results are treated. They are presented in a unifying style, being accompanied by definitions, preliminary results and sometimes by examples, as this work aims

to become a useful reference for potential doctoral students.

The work is organized in eight chapters and three appendices.

In Chapter 1 we first define the differential structures on locally Euclidean spaces and recall several splitting classical results of manifolds which are used later to obtain some necessary conditions on a map to have finite critical set. In the first section we prove several facts involving critical sets and their images both in the differential category as well as in the topological one. Among them we emphasize the inclusion of the critical set of the product of two maps, with values in a Lie group, in the cross product of the critical sets of the two factors. Classical facts on differentiable forms, most of which are subsequently applied in this work, are also part of the first section alongside with some more recent proved results. Section 1.2 is devoted to the degree of a differentiable map between two compact orientable differential manifolds and one application towards a global injectivity result. In the next section we recall classical results in the regular and critical settings such as the *preimage theorem*, the *Ehresmann Theorem*, various versions of the *Sard Theorem*, an embedding theorem, the *Minimax Principle* and the *Lusternik-Schnirelmann Multiplicity Theorem*. The preimage theorem and the Ehresmann theorem are accompanied by several examples. The class of Montgomery-Samelson fiberings, between two manifolds M and N, is extended to the class $CS^{\infty}(M, N)$ consisting in those functions which separate the critical values from the regular ones.

Chapter 2 is devoted to transversality and some of its applications. In its first section we define the transversal intersection of a differentiable map with a submanifold of the target manifold and point out several results, most of which are classical. However one proved result concerns the possibility to deform one immersion differentiably in such a way that the immersablity is preserved all the way and the range of the final immersion avoids a given finite set of the target manifold. The regularity of a point in the source manifold of a composed function is characterized, in Section 2.2, through some transversal behavior. One application of this characterization is the method of Lagrange multipliers in the finite dimensional setting which is extensively treated both theoretically and through one example. This method is also exploited in the infinite dimensional setting through one example of an eigenvalue problem which is treated through a minimization procedure. In Section 2.4 the critical points of a certain restriction to a hypersurface are realized as characteristic points of that hypersurface with respect to an involutive distribution. The characeristic points are then extended to tangency points and several results on the tangency set are then proved. The last section is devoted to the order

of connectedness of several inclusion maps and the size of the vanishing set of top differentiable forms on non-orientable manifolds.

Chapter 3 is devoted to classical results on functions with isolated cone-like singularities, such as some *polynomial functions*, *Lefschetz fibrations*, *Morse functions*. The *vanishing cycles* associated to Lefschetz fibrations are generalized and described in a more general setting. Based on generalized Neuwirth-Stallings pairs, we provide, in Section 3.3, some constructions of fibered links in dimensions $(2n, n)$, $n \geq 3$, $(2n+1, n)$, $n \geq 2$ and $(2n, k), (2n+1, k), n \geq k \geq 2$.

In Chapter 4 we emphasize a few types of closed subsets of a manifold M which are not N-critical, i.e. they are not critical sets of any map $f : M \longrightarrow N$. In this respect we recall the definitions of the φ-category of a pair (M, N) of manifolds, as the finite subsets of M with cardinality smaller than this category are not N-critical. The special cases of Morse functions and $N = \mathbb{R}$ or S^1 leads us to the so called Morse-Smale characteristics (or shortly the *M-S characteristics*) of a manifold M which are also defined and studied. Note that the finite subsets of M with cardinality smaller than the real or circular category are not critical for any real or circular Morse functions on M, respectively. The real Morse-Smale characteristic is related to the minimum number of cells in the CW-decompositions of M up to homotopy. It is also a lower bound for the total curvature of M with respect to its embeddings in Euclidean spaces. The first two sections are devoted to the φ-category of a pair of manifolds and the Morse-Smale categories of a manifold. These categories are explicitly computed in several particular cases.

In Section 4.3, some information on the homology groups of a manifold with an infinite closed countable set removed are also pointed out. Based on this information we provide, in the last section, some sufficient conditions on the base space, the total space and on the fiber of a fibration, of topological nature, which ensure the CS^∞-non-criticality of some collections of countably many fibers.

Chapter 5 is devoted to applications with a finite number of critical points. In Section 5.1 we describe the regular fiber of some maps with a finite number of critical points between two manifolds which are subject to some topological conditions. Some preparatory results towards a lower estimate of some φ-category are also pointed out. We also review some much deeper necessary conditions on two manifolds M, N in order to have finite $\varphi(M, N)$. Some of them are also sufficient and for them the φ-category is either computed or some estimates are provided. Section 5.2 deals with some examples of pairs of manifolds with finite φ-category. The construction of these manifolds is based, at first, on a given bicolored graph whose black vertices are decorated with

local models of isolated singularities and each white vertex is decorated with a manifold whose boundary has as many connected components as the degree of the vertex itself. An identification procedure which depends on the given bicolored decorated graph gives rise to a manifold with boundary which admits a map with a finite number of critical points over a certain disk. The singularities correspond to the black vertices and the restriction to the boundary is a locally trivial fibration. Working with several bicolored decorated graphs $\Gamma_1, \ldots, \Gamma_p$, whose restrictions to the boundaries of their associated manifolds are cobounded fibrations, gives rise to a boundaryless manifold $M(\Gamma_1, \ldots, \Gamma_p)$ which admits a map with finitely many critical points over a sphere S^n. Some estimates for the φ-category of the pair $(M(\Gamma_1, \ldots, \Gamma_p), S^n)$ are provided.

Chapter 6 is devoted to pairs of manifolds with infinite φ-category. Such pairs of manifolds are selected through their topology (Section 6.1), via their homology and/or their homotopy groups (Section 6.2). For some of them we actually prove that every map between them has high dimensional critical set (Sections 6.1 and 6.3). This is the case, for example, when $\pi_1(M^n)$ is finite, $\pi_1(N^n)$ is infinite and $n \geq 2$. Using an approach of geometric flavor we recapture some of these results and prove other different ones. For several pairs of manifolds we are only able to prove, in Sections 6.2 and 6.3, that all functions between them have infinitely many critical points.

In Chapter 7 we first recall the link between the critical points and characteristic points. A hypersurface Σ of the source manifold of a real or S^1-valued submersion f is needed to realize this connection. Indeed the critical points of the restriction $f\big|_\Sigma$ are the characteristic points of Σ with respect to the involutive distribution of the tangent hyperplanes to the fibers of f. In other words, the theory of characteristic and tangency sets of hypersurfaces with respect to arbitrary distributions fits well in this context of critical points and critical sets. In Section 7.1 some estimates on the minimum number of characteristic points of the compact orientable surface $\Sigma_g \subset \mathbb{R}^3$ of genus g with respect to the horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$ are provided. In Section 7.2 we provide general lower and upper bounds for the Hausdorff dimension of the tangency set of a submanifold of $\mathbb{R}^{n+m}$ with respect to a distribution of rank n. Some special attention is paid to the particular case of the highly non-involutive horizontal distribution of the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$. As a consequence we generalize, in the last section, the Derridj's theorem.

In Chapter 8 an elliptic eigenvalue-transmission problem (or E-T problem, shortly) with Neumann boundary conditions is analyzed. The involved elliptic operators are the p- and q-Laplacians and the problem is tackled variationally.

In Section 8.1 the variational version of the problem is described alongside with the appropriate functionals, which are shown to verify the hypothesis of the *Lusternik-Schnirelmann principle* (shortly the *L-S principle*). The same functionals were considered in Chapter 2 to treat a constrained minimization problem which is shown there to have nonempty solution set. It turns out that the solutions of that minimization problem are, via the Lagrange multiplier rule, solutions of the elliptic eigenvalue-transmission problem. On the other hand, the L-S principle produces an unbounded sequence of positive eigenvalues and the minimization procedure produces infinitely many eigenfunctions of the initial eigenvalue-transmission problem. In Section 8.2 an eigenvalue-transmission problem with Robin boundary conditions is analyzed and the variational version of this new problem is described alongside with the appropriate functionals. The counterparts of the results in Section 8.1 can be proved in a similar manner. Finally, in Section 8.3 an eigenvalue-transmission problem in the Riemannian setting is analyzed and the counterparts of the results in Section 8.1, which can be similarly proved, are stated.

At the end of the book we included three appendices on *Banach spaces and differential calculus* (A), *Various dimensions of a space* (B) and *Homotopy Theory* (C) as we tried to make it self-contained. However we provide references whenever we use a result not mentioned here.

Chapter 1

The sets of critical points and critical values

In this Chapter we first define the differential structures on locally Euclidean spaces and recall some classical splitting results for 3,6, and higher dimensional manifolds. These results are used in Section 5.1 to obtain some necessary conditions on a map to have finite critical set. We close the first section by proving several facts involving the critical sets and the sets of critical values both in the category of topological manifolds and continuous maps and the category of differential manifolds and differential maps. Section 1.2 is devoted to the degree of a differentiable map between two compact orientable differentiable manifolds and one of its applications. This application concerns a global injectivity result which is based both on the degree of some radial projection and a separation result. In the last section we recall some classical results in the regular and critical settings which are subsequently used in this work. Here we recall the *preimage theorem*, the *Ehresmann Theorem*, various versions of the *Sard Theorem*, an embedding theorem, the *Minimax Principle* and the *Lusternik-Schnirelmann Multiplicity Theorem*. Some applications of the Ehresmann Theorem and of the Sard Theorem are also pointed out. In the context of the Ehresmann theorem we define and study the class $CS^\infty(M, N)$, which contains the Montgomery-Samelson fiberings, consisting in those maps which separate the critical values from the regular ones.

Chapter 2

Transversality and applications

In this chapter we first recall a few classical results and prove some more recent ones, involving the transversality. The Lagrange multiplier rule, in the finite dimensional setting, appears as an application of such a proved result which characterizes the regular set of a composed function. The rule of Lagrange multipliers, in the infinite dimensional framework, is stated, and an application for each of the two Lagrange multiplier rules is highlighted. The first application concerns the convex regular levels of a polynomial function. The second application concerns the constrained minimum value of one integral functional defined on certain subspace of the product of two Sobolev spaces. The constraint is the intersection of two level sets of some other two integral functionals.

In section 2.4 of this chapter we first provide an interpretation of some critical points as characteristic points. This interpretation concerns the critical points of the restriction of a real or S^1-valued function to a hypersurface of the Euclidean space, which end up as characteristic points of that hypersurface with respect to an involutive distribution. The characteristic points are then extended to tangency points with respect to arbitrary distributions. Several results on the tangency set of a submanifold of the Euclidean space, with respect to the highly non-involutive distributions which fulfill the Hörmander condition, are then proved.

The last section concerns the order of connectedness of the inclusion maps of some complements depending on the set of critical values of a differentiable map. The vanishing sets of top-differentiable forms on non-orientable manifolds are also shown to be at least 1-dimensional.

Chapter 3

Functions with isolated cone-like critical points

This chapter is devoted to classical results on functions with isolated cone-like singularities, such as some *polynomial functions*, *Lefschetz fibrations*, *Morse functions*. The *vanishing cycles* associated to Lefschetz fibrations are described in a more general setting. Based on generalized Neuwirth-Stallings pairs, we provide, in Section 3.3, some constructions of fibered links in dimensions $(2n, n)$, $n \geq 3$, $(2n+1, n)$, $n \geq 2$ and $(2n, k), (2n+1, k), n \geq k \geq 2$. These constructions are based on links of S^{n-1}-spheres into S^{2n-1} which are uniquely determined, up to isotopy, according to Haefliger's classification theorem (see [87, 88]), by their linking matrices. The *generalized Hopf links* are defined by particular linking matrices and the construction of fibered links in dimensions $(2n, n)$, $n \geq 3$ are based on such links. A spinning procedure applied to the fibered links already constructed before gives rise to the fibered links in dimensions $(2n+1, n)$, $n \geq 2$. Finally, by a projection procedure, one can decrease the dimension of target sphere of an existing fibered link. Such a procedure gives rise to the fibered links in dimensions $(2n, k), (2n+1, k), n \geq k \geq 2$.

3.1 Cone-like critical points

Definition 3.1.1. Let $V = f^{-1}(f(x))$, where x is a critical point of $f : N_1^{n+q} \longrightarrow N_2^n$. Following King (see [107] and [75]), the critical point x is called *cone-like* if it admits a cone neighborhood in V, i.e. there exists some closed manifold $L \subset V \setminus \{x\}$ and a neighborhood N of x in V which is homeomorphic to the cone $C(L)$ over L. Recall that the cone is defined as the quotient $C(L) = L \times (0,1]/L \times \{1\}$. Then the manifold L is called the *local link* at x. If x is

Chapter 4

The inverse critical point problem

We have already seen in the first chapter that the critical sets are all closed subsets of the source manifolds. The inverse critical set problem consists in deciding whether or not a given closed subset of a certain manifold is a critical set for a differentiable map defined on that manifold.

We can specialize this problem by restricting the whole class of differentiable maps between two manifolds to some of its special subclasses. A closed subset C of M is called $\mathcal{F}$-*critical*, where $\mathcal{F} \subseteq C^\infty(M, N)$, if $C = C(f)$ for some differentiable map $f \in \mathcal{F}$. A $C^\infty(M, N)$-critical set will be called N-*critical* and an $\mathbb{R}$-critical set will be simply called *critical*. Given a closed subset C of M, the question is: *is it* $\mathcal{F}$-*critical ?* This is a fundamental problem which has been treated in [86] and [150] for $\mathcal{F} = C^\infty(M, \mathbb{R})$ and its subfamily of smooth proper functions. For instance the *Antoine's Necklace* of $\mathbb{R}^3$ is a properly critical set [86] while the circle $S^1 \subseteq \mathbb{R}^2$ is not a critical set [150]. On the other hand the finite subsets of M having cardinality strictly smaller then the Lusternick-Schnirelmann category of M, are not critical if M is compact, because every function $f : M \to \mathbb{R}$ has at least $\operatorname{cat}(M)$ critical points (see Theorem 1.3.16). Let us also mention that the S^m-non-criticality of certain subsets of M, provides information on the set of zeros of the Gauss-Kronecker curvature associated to an arbitrary immersion of M into $\mathbb{R}^{m+1}$, whenever M^m is immersible into $\mathbb{R}^{m+1}$, taking into accout that the mentioned set of zeros is actually the critical set of the associated Gauss map of the given immersion. For instance the product $S^k \times S^n$, $k + n \geq 3$ has, for any immersion $f : S^k \times S^n \to \mathbb{R}^{k+n+1}$, infinitely many points of zero Gauss-Kronecker curvature, simply because the finite subsets of $S^k \times S^n$ are not S^{k+n}-critical [161].

In fact, the topic of functions with infinitely many critical points is intensively studied in the chapter 6 of this work.
In the first section we first review the φ-category of some pairs of surfaces, completely computed for the involved pairs and compute the φ-category of some connected sums of a few products of spheres. In Section 4.2 we point out several general estimates on the real and circular Morse-Smale characteristics (or shortly the M-S characteristic). For some particular manifolds, the relation between the two characteristics and complete computations for some other manifolds are also provided. We also compute the circular Morse-Smale characteristic of the compact surfaces.
We next provide information on the homology groups of a manifold M^n with an infinite closed countable set removed. For example it is proved here that the complement of a closed countable subset of a given n-dimensional boundaryless manifold has large $(n-1)$-homology group. Based on this information we finally point out some sufficient conditions on the base space, the total space and on the fiber of a fibration, of topological nature, which ensure the CS^{∞}-non-criticality of some collections of countably many fibers.

4.1 The φ-category of a pair of manifolds. Examples

In addition to the above examples of finite non-critical sets (i.e. those with the cardinal smaller than the Lusternick-Schnirelmann category of the ambient manifold), we also encountered some other examples of finite critical or non-critical sets. For instance, some finite subsets of a compact manifold are critical sets for the Morse functions defined on that manifold. On the other hand, finite subsets of compact manifolds are non-critical in several situations for maps $f : M^m \to N^n$. One such situation appears when $m = n \geq 2$ and M^n and N^n are compact manifolds topologically away from each other (see Proposition 1.1.10). Another situation appears when $m \geq n \geq 2$ and M is compact while N is connected and non-compact (see Proposition 6.3.2). The non-criticality of finite subsets of a manifold M for maps $f : M \longrightarrow N$ can be symbolized by $\varphi(M, N) = \infty$, where $\varphi(M, N)$ is the so called *φ-category* of the pair (M, N), which is, otherwise, defined by

$$\varphi(M, N) := \min\{\text{card}(C(f))\ f \in C^{\infty}(M, N)\}.$$

Note that $\varphi(M, N) = 0$ if and only if $C^{\infty}(M, N)$ contains immersions, submersions or local diffeomorphisms, depending on the order of the dimensione of the two manifolds.

Chapter 5

Maps with finite critical sets

This chapter is devoted to maps with a finite number of critical points. In Section 5.1 we describe the regular fiber of some maps with a finite number of critical points between two manifolds that are subject to certain topological conditions. We also review a few much deeper necessary conditions on two manifolds M, N in order to have finite $\varphi(M, N)$. Some of them are also sufficient and φ-category is either computed or some estimates are provided under such necessary and sufficient conditions. Section 5.2 deals with an example of pairs of manifolds with finite φ-category. The construction of this manifold is based, at first, on a given bicolored graph whose black vertices are decorated with local models of isolated singularities and each white vertex is decorated with a manifold whose boundary has as many connected components as the degree of the vertex itself. The local model of a fiberd link (see section 3.3) is first defined in Section 5.2. An identification procedure which depends on the given bicolored decorated graph gives rise to a manifold with boundary which admits a function with a finite number of critical points over a certain disk, whose singularities correspond to the black vertices and whose restriction to the boundary is a locally trivial fibration. Working with several bicolored decorated graphs $\Gamma_1, \dots, \Gamma_p$, whose restrictions to the boundaries of their associated manifolds are cobounded fibrations, gives rise to a boundaryless manifold $M(\Gamma_1, \dots, \Gamma_p)$ which admits a function with a finite number of critical points over a sphere S^n. Some estimates for the φ-category of the pair $(M(\Gamma_1, \dots, \Gamma_p), S^n)$ are provided and for some particular bicolored decorated graphs, with one single black vertex, this φ-category is shown to be one.

Chapter 6

Maps with infinite critical sets

This chapter is devoted to pairs of manifolds with infinite φ-category. Such pairs of manifolds are selected through their topology and orientability (Section 6.1) as well as via their homology and/or their homotopy groups (Section 6.2). For some of them we prove that every map between them has high dimensional critical set (Sections 6.1 and 6.3). This is the case, for example, when $\pi_1(M^n)$ is finite, $\pi_1(N^n)$ is infinite and $n \geq 2$. Using an approach of geometric flavor we recapture some of these results and prove a few more as well.

6.1 The zero codimension case

In order to justify the main result of this section, inspired by ([165]), we need the following:

Theorem 6.1.1. ([*101*]) *Every compact connected manifold M^m is a Cantor manifold. More precisely, no subset of M of dimension $\leq n-2$ separates M, where $n = \dim M$. Consequently, every subset of M which separates M has dimension at least $n-1$, i.e. no subset of M of dimension $\leq m-2$ separates M.*

Theorem 6.1.2. ([*165*]) *Let M^n, N^n, $n \geq 2$ be smooth orientable manifolds. If M is compact and $f : M \to N$ has zero degree, then either $C(f) = M$ or the set $R(f) = M \backslash C(f)$ is not connected.*

Corollary 6.1.3. *Let M^n, N^n, $(n \geq 2)$ be smooth orientable manifolds. If M is compact and $f : M \to N$ has zero degree, then $\dim[C(f)] \geq n-1$.*

Chapter 7

The size of the tangency set

Let M^m be a manifold and let f be a real-valued or S^1-valued smooth submersion on M. If $\Sigma \subset M$ is a hypersurface, then $x \in \Sigma$ is, according to Remark 2.4.1, a critical point of the restriction $f\big|_\Sigma$ if and only if $T_x(\Sigma) = \ker(df)_x$. Thus, the critical points of $f\big|_\Sigma$ are the characteristic points of the hypersurface Σ with respect to the involutive distribution $\{\ker(df)_x\}_{x\in M}$ of the tangent hyperplanes to the fibers of f. In other words the *characteristic point* looks like an extended concept for the *critical point* of real or S^1 valued functions. The extension is given by the extended class of distributions we may consider, when working with characteristic points, to arbitrary distributions, while the critical points appear in relation with some particular involutive distributions. Recall that the characteristic points are further generalized to tangency points, which are defined for a submanifold of arbitrary dimension $n \leq m$ relative to a distribution of equal or higher rank than n. In other words, the tangency points and tangency sets of a submanifold of some Euclidean space with respect to a given distribution, possibly non-involutive, fits well in this context of critical points and critical sets. In fact we provide some information on the size of the tangency set of an n-dimensional submanifold of $\mathbb{R}^{n+m}$ with respect a given distribution of rank n on $\mathbb{R}^{n+m}$. Some special attention is paid to the particular case of the non-involutive horizontal distribution of the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$. In Section 7.1 some estimates on the minimum number of characteristic points of the compact orientable surface $\Sigma_g \subset \mathbb{R}^3$ of genus g with respect to the horizontal distribution of the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$ are provided. In Section 7.2 a general upper estimate for the Hausdorff dimension of the tangency set of of a submanifold of $\mathbb{R}^{n+m}$ with respect to a distribution of rank n is given. In the last section of the chapter we first prove this upper estimate and generalize the Derridj's theorem.

Chapter 8

BVP and critical points of appropriate energy functionals

In this chapter we present several boundary value problems (or shortly BVP), some of whose solutions can be obtained by minimizing appropriate functionals on suitable Sobolev spaces. In the first section we present the details of a particular BVP and review several others along with their appropriate energy functionals. In the remaining sections we describe and analyse, following [25], a few elliptic eigenvalue-transmission problems (or shortly E-T problems). The involved operators are the $p-$ and $q-$Laplacians and the boundary conditions have Neumann or Robin flavor. The problems are approached variationally through the L-S (Lusternick-Schnirelmann) principle in Banach spaces.

8.1 A few BVP and a particular eigenvalue-transmission problem

• The solutions in $H_0^1(\Omega)$ of the Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{p-1} + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a smooth bounded domain, $p > 1$ and h is a given function on $L^2(\Omega)$, are the critical points of the functional

$$I : H_0^1(\Omega) \longrightarrow \mathbb{R},\ I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx - \int_\Omega hu dx.$$

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